Lesson 19. Optimization with Equality Constraints, cont.

1 Overview

- Last time: the Lagrange multiplier method for optimization problems with one equality constraint
- Today: multiple equality constraints

2 The Lagrange multiplier method – k equality constraints

$$\begin{aligned} \text{minimize/maximize} \quad & f(x_1, \dots, x_n) \\ \text{subject to} \quad & g_1(x_1, \dots, x_n) = c_1 \\ & \vdots \\ & g_k(x_1, \dots, x_n) = c_k \end{aligned}$$

- The **Lagrangian function** L is

$$L(\lambda_1, \dots, \lambda_k, x_1, \dots, x_n) = f(x_1, \dots, x_n) - \lambda_1\big[g_1(x_1, \dots, x_n) - c_1\big] - \dots - \lambda_k\big[g_k(x_1, \dots, x_n) - c_k\big]$$

- The gradient of L is

$$\nabla L(\lambda_1, \dots, \lambda_k, x_1, \dots, x_n) = \begin{bmatrix} -g_1(x_1, \dots, x_n) + c_1 \\ \vdots \\ -g_k(x_1, \dots, x_n) + c_k \\ \frac{\partial f}{\partial x_1}(x_1, \dots, x_n) - \lambda_1 \frac{\partial g_1}{\partial x_1}(x_1, \dots, x_n) - \dots - \lambda_k \frac{\partial g_k}{\partial x_1}(x_1, \dots, x_n) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_1, \dots, x_n) - \lambda_1 \frac{\partial g_1}{\partial x_n}(x_1, \dots, x_n) - \dots - \lambda_k \frac{\partial g_k}{\partial x_n}(x_1, \dots, x_n) \end{bmatrix}$$

- The Hessian of L is

$$H_L(\lambda_1, \dots, \lambda_k, x_1, \dots, x_n) = \begin{bmatrix} 0 & \dots & 0 & -\frac{\partial g_1}{\partial x_1}(x_1, \dots, x_n) & \dots & -\frac{\partial g_1}{\partial x_n}(x_1, \dots, x_n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & -\frac{\partial g_k}{\partial x_1}(x_1, \dots, x_n) & \dots & -\frac{\partial g_k}{\partial x_n}(x_1, \dots, x_n) \\ -\frac{\partial g_1}{\partial x_1}(x_1, \dots, x_n) & \dots & -\frac{\partial g_k}{\partial x_1}(x_1, \dots, x_n) & h_{11} & \dots & h_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial g_1}{\partial x_n}(x_1, \dots, x_n) & \dots & -\frac{\partial g_k}{\partial x_n}(x_1, \dots, x_n) & h_{n1} & \dots & h_{nn} \end{bmatrix}$$

where

$$h_{ij} = \frac{\partial^2 f}{\partial x_j \partial x_i}(x_1, \dots, x_n) - \lambda_1 \frac{\partial^2 g_1}{\partial x_j \partial x_i}(x_1, \dots, x_n) - \dots - \lambda_k \frac{\partial^2 g_k}{\partial x_j \partial x_i}(x_1, \dots, x_n)$$

Finding constrained local optima:

- **Step 0.** Form the Lagrangian function L and find its gradient and Hessian
- **Step 1.** Find the **constrained critical points** $(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_n)$ by solving the system of equations:

$$\nabla L(\lambda_1, \ldots, \lambda_k, x_1, \ldots, x_n) = 0$$

- **Step 2.** Classify each constrained critical point as a local minimum, local maximum, or saddle point by applying the **second derivative test for constrained extrema**:
    - Suppose $(\lambda_1^*, \ldots, \lambda_k^*, x_1^*, \ldots, x_n^*)$ is a constrained critical point found in Step 1
    - Compute the principal minors $d_i = |H_L(\lambda_1^*, \ldots, \lambda_k^*, x_1^*, \ldots, x_n^*)|$ for $\underline{i = 2k+1, \ldots, n+k}$
    - If $d_{n+k} \neq 0$:

      (1) $(-1)^k d_{2k+1} > 0, (-1)^k d_{2k+2} > 0, \ldots, (-1)^k d_{n+k} > 0$ then f has a constrained local minimum at $(x_1^*, \ldots, x_n^*)$

      (2) $(-1)^k d_{2k+1} < 0, (-1)^k d_{2k+2} > 0, (-1)^k d_{2k+3} < 0, \ldots$ then f has a constrained local maximum at $(x_1^*, \ldots, x_n^*)$

      (3) otherwise, f has a constrained saddle point at $(x_1^*, \ldots, x_n^*)$
    - If $d_{n+1} = 0$, then the test gives no information

Example 1. Use the Lagrange multiplier method to find the local optima of

$$\begin{aligned} \text{minimize/maximize} \quad & x_3 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 12 \\ & x_1^2 + x_2^2 - x_3 = 0 \end{aligned}$$

- In this problem, $n =$ ☐ and $k =$ ☐

Step 0. Form the Lagrangian function L and find its gradient and Hessian.

Step 1. Find the constrained critical points.

Step 2. Classify the constrained critical points as a constrained local minimum, constrained local maximum, or constrained saddle point.

3 Exercises

Problem 1. Use the Lagrange multiplier method to find the local optima of

$$\begin{aligned} \text{minimize/maximize} \quad & x_1^2 + x_2^2 + x_3^2 \\ \text{subject to} \quad & 3x_1 + x_2 + x_3 = 5 \\ & x_1 + x_2 + x_3 = 1 \end{aligned}$$